c) $0 < m_4 \leqslant B(\beta)$ and $R_i(\beta) \leqslant m_5$ on $[0, b]$.

Dome-shaped shells whose middle surface is a part of an ellipsoid, paraboloid, two-sheet hyperboloid, and other surfaces of revolution.

2) According to [4] condition (2.2) guarantees the convergence of projective methods.

3) Similar results can be obtained in the case of a shallow symmetrically loaded spherical dome and other shallow symmetrically loaded surfaces of revolution ($\psi = w' B^{-1}$ in (1.1)), if conditions (a), (b), (c) are satisfied.

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CAUCHY PROBLEM FOR A VISCOELASTIC TRANSVERSELY ISOTROPIC MEDIUM

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We solve a Cauchy problem for a viscoelastic transversely isotropic medium. Generalizing the method of separation of variables for certain classes of static problems treated in [1], and using this method, we reduce the system of integro-partial differential equations to a system of ordinary differential equations in the time coordinate. Solving the latter system by the method of averaging [2, 3], we obtain explicit formulas characterizing the propagation of waves in a viscoelastic transversely isotropic medium.

Using the relationship between stress and strain for the medium in question [4] and identifying the regular part of the relaxation kernels, we write the system of equations for a viscoelastic transversely isotropic medium in cylindrical coordinates as follows:

$$
\begin{aligned}
&(c_{66}-c_{66}^*)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\pm\frac{2i}{r^2}\frac{\partial}{\partial\varphi}-\frac{1}{r^2}\right)W_{1,2}+ \\
&(c_{55}-c_{55}^*)\frac{\partial^2 W_{1,2}}{\partial z^2}+\left(\frac{c_{11}+c_{12}}{4}-\frac{c_{11}^*+c_{12}^*}{4}\right)\left(\frac{\partial}{\partial r}\pm\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\times \\
&\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_1+\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]+ \\
&[(c_{13}+c_{55})-(c_{13}^*+c_{55}^*)]\left(\frac{\partial}{\partial r}\pm\frac{i}{r}\frac{\partial}{\partial\varphi}\right)\frac{\partial W_3}{\partial z}=\rho\frac{\partial^2 W_{1,2}}{\partial t^2}
\end{aligned} \tag{1}
$$

$$(c_{55}-c_{55}^*)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)W_3+(c_{33}-c_{33}^*)\frac{\partial^2 W_3}{\partial z^2}+$$
$$\left(\frac{c_{13}+c_{55}}{2}-\frac{c_{13}^*+c_{55}^*}{2}\right)\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_1+\right.$$
$$\left.\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\varphi}+\frac{1}{r}\right)W_2\right]=\rho\frac{\partial^2 W_3}{\partial t^2}$$

$$(W_{1,2}=u_r\pm iu_\varphi,\quad W_3=u_z)$$

$$c_{12}^*\varphi(t)=\int_0^t\Gamma_1\varphi(\tau)\,d\tau,\qquad c_{66}^*\varphi(t)=\int_0^t\Gamma_2\varphi(\tau)\,d\tau$$

$$c_{13}^*\varphi(t)=\int_0^t\Gamma_3\varphi(\tau)\,d\tau,\qquad c_{33}^*\varphi(t)=\int_0^t\Gamma_4\varphi(\tau)\,d\tau$$

$$c_{55}^*\varphi(t)=\int_0^t\Gamma_5\varphi(\tau)\,d\tau,\qquad c_{11}^*\varphi(t)=\int_0^t(\Gamma_1+2\Gamma_2)\,\varphi(\tau)\,d\tau$$

Here the c_{ij} are the elastic constants of the given medium; the c_{ij}^* are operators defined by the formulas given above; $\Gamma_i=\Gamma_i(t-\tau)$ are the regular parts of the relaxation kernels (for brevity we shall henceforth omit the argument $t-\tau$).

For reinforced load-carrying structures made of polymer materials the regular parts of the relaxation kernels are proportional to small parameters [4], therefore in the system (1) we can replace the c_{ij}^* by εc_{ij}^* ($\varepsilon>0$ is a small parameter). The method of averaging, used in solving the problem, the existence of the small parameter ε. In the final results we can set $\varepsilon=1$.

We seek a solution of system (1) in the form

$$W_l=J_\nu(\alpha r)e^{i(k\varphi+\beta z)}\;T_l(t),\qquad l=1,2,3,\quad \nu=k\pm1,\;k \tag{2}$$

where the $J_\nu(\alpha r)$ are Bessel functions of the first kind. After substituting the expressions (2) into Eq. (1) we obtain a system of ordinary integro-differential equations in the time coordinate

$$\rho T''_{1,2}+\left(\frac{c_{11}+c_{66}}{2}\alpha^2+c_{55}\beta^2\right)T_{1,2}-\frac{c_{11}+c_{12}}{4}\alpha^2 T_{2,1}\pm \tag{3}$$
$$(c_{13}+c_{55})\,i\beta\alpha T_3=\varepsilon\int_0^t\left[\left(\frac{\Gamma_1+3\Gamma_2}{2}\alpha^2+\Gamma_5\beta^2\right)T'_{1,2}(\tau)-\right.$$
$$\left.\frac{\Gamma_1+\Gamma_2}{2}\alpha^2 T'_{2,1}(\tau)\pm(\Gamma_3+\Gamma_5)\,i\beta\alpha T'_3(\tau)\right]d\tau$$

$$\rho T_3''+(c_{55}\alpha^2+c_{33}\beta^2)T_3-\frac{c_{13}+c_{55}}{2}i\beta\alpha(T_1-T_2)=$$
$$\varepsilon\int_0^t\left\{(\Gamma_5\alpha^2+\Gamma_4\beta^2)T_3(\tau)-\frac{\Gamma_3+\Gamma_5}{2}i\beta\alpha\,[T_1(\tau)-T_2(\tau)]\right\}d\tau$$

The solution of the system (3) when the viscosity (the integral term) is not taken into account has the form

$$T_{1,2}(t)=C_1\cos\lambda_1 t+C_2\sin\lambda_1 t\pm C_3\cos\lambda_2 t\pm C_4\sin\lambda_2 t\pm C_5\cos\lambda_3 t\pm C_6\sin\lambda_3 t \tag{4}$$
$$T_3(t)=a_2(C_3\cos\lambda_2 t+C_4\sin\lambda_2 t)+a_3(C_5\cos\lambda_3 t+C_6\sin\lambda_3 t)$$

$$a_l = \frac{-\rho\lambda_l^2 + c_{11}\alpha^2 + c_{55}\beta^2}{(c_{13}+c_{55})\, i\beta\alpha}, \quad \lambda_1 = \left[\frac{1}{\rho}\,(c_{66}\alpha^2 + c_{55}\beta^2)\right]^{1/2}$$

$$\lambda_{2,3} = \left\{\frac{1}{2\rho}\left[(c_{55}\alpha^2 + c_{33}\beta^2) + (c_{11}\alpha^2 + c_{55}\beta^2) \mp R\right]\right\}^{1/2}$$

$$R = \{[(c_{55}\alpha^2 + c_{33}\beta^2) - (c_{11}\alpha^2 + c_{55}\beta^2)]^2 + 4\,(c_{13}+c_{55})^2\alpha^2\beta^2\}^{1/2}$$

To find the solution of the system (3) with viscosity taken into account we reduce the system to "standard form" by applying the method of variation of parameters in which the C_i are considered as unknown functions of time

$$C_{1,2}^{\cdot} = \mp\frac{\varepsilon}{\lambda_1}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_1 t F_{11}(C_1,\, C_2) \tag{5}$$

$$C_{3,4}^{\cdot} = \mp\frac{\varepsilon}{\lambda_2 m_2}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_2 t\,[F_{22}(C_3,\, C_4) + F_{23}(C_5,\, C_6)]$$

$$C_{5,6}^{\cdot} = \mp\frac{\varepsilon}{\lambda_3 m_3}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_3 t\,[F_{32}(C_3,\, C_4) + F_{33}(C_5,\, C_6)]$$

Here

$$F_{ij}(C_l,\, C_{l+1}) = \int_0^t \Gamma_{ij}(t-\tau)(C_l\cos\lambda_j\tau + C_{l+1}\sin\lambda_j\tau)\,d\tau$$

$$m_{2,3} = \pm(a_3 - a_2), \quad \Gamma_{11}(t-\tau) = \Gamma_2\alpha^2 + \Gamma_5\beta^2$$

$$\Gamma_{22,33} = (\Gamma_5\alpha^2 + \Gamma_4\beta^2)\, a_{2,3} - [(\Gamma_1 + 2\Gamma_2)\,\alpha^2 + \Gamma_5\beta^2]\; a_{3,2} - 2\,(\Gamma_3 + \Gamma_5) i\beta\alpha$$

$$\Gamma_{23,32} = [(\Gamma_5 - \Gamma_1 - 2\Gamma_2)\,\alpha^2 + (\Gamma_4 - \Gamma_5)\,\beta^2]a_{3,2} - (1 + a_{3,2}^2)\;(\Gamma_3 + \Gamma_5)\; i\beta\alpha$$

are unknown quantities.

To the system (5) we make correspond the averaged system

$$\xi_{2j-1}^{\cdot} = -\frac{\varepsilon}{2\lambda_j m_j}\,(B_{0j}\xi_{2j-1} + A_{0j}\xi_{2j}) \tag{6}$$

$$\xi_{2j}^{\cdot} = \frac{\varepsilon}{2\lambda_j m_j}\,(A_{0j}\xi_{2j-1} - B_{0j}\xi_{2j}), \quad j = 1, 2, 3, \quad m_1 = 1$$

Here, as is easily shown to be the case [2–4]

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_i t\int_0^t \Gamma_{ij}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\lambda_i\tau d\tau = \begin{cases}\pm{}^1/_2\, B_{0i}, & i = j\\ 0, & i \neq j\end{cases}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_i t\int_0^t \Gamma_{ij}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\lambda_j\tau d\tau = \begin{cases}{}^1/_2\, A_{0i}, & i = j\\ 0, & i \neq j\end{cases}$$

$$A_{0i} = \int_0^\infty \Gamma_{ii}(\tau)\cos\lambda_i\tau d\tau, \quad B_{0i} = \int_0^\infty \Gamma_{ii}(\tau)\sin\lambda_i\tau d\tau$$

$$A_{01} > 0, \; A_{02} < 0, \; A_{03} > 0, \; B_{01} > 0, \; B_{02} < 0, \; B_{03} > 0$$

Integrating the system (6), we obtain

$$\xi_{2j-1,\,2j}(t) = \exp\left(-\frac{\varepsilon B_{0j}}{2\lambda_j m_j}\,t\right)\left(A_j\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\frac{\varepsilon A_{0j}}{2\lambda_j m_j}\,t \mp B_j\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\frac{\varepsilon A_{0j}}{2\lambda_j m_j}\,t\right)$$

where the A_j and B_j are arbitrary constants, $j = 1,2,3$. It was shown in [3], under very general conditions, that over a sufficiently large finite time interval the solution of the system (6) is arbitrarily close to the solution of the system (5).

In accord with theorems on averaging the solution of the system (3) can be represented approximately in the form (4) wherein the C_j are replaced by $\xi_j(t)$. The most general form of the solution is obtained by substituting this approximate solution into Eq. (2), summing with respect to k and integrating with respect to α and β

$$W_l(r, \varphi, z, t) = \sum_{k=-\infty}^{\infty} e^{ik\varphi} \int_0^{\infty} J_\nu(\alpha r)\, d\alpha \int_0^{\infty} e^{i\beta z} \times \tag{7}$$

$$\sum_{i=1}^{3} n_{li} \exp\left(-\frac{\varepsilon B_{0i}}{2\lambda_i m_i} t\right) [A_i \cos \mu_i t + B_i \sin \mu_i t]\, d\beta$$

$$\mu_i = \lambda_i - \frac{\varepsilon A_{0i}}{2\lambda_i m_i}, \quad l = 1, 2, 3, \quad \nu = k \pm 1,\ k$$

$$n_{1i} = n_{21} = 1, \quad n_{22} = n_{23} = -1, \quad n_{31} = 0, \quad n_{32} = a_2, \quad n_{33} = a_3$$

A comparison of the solution obtained with that given in [5] shows that in the case of a viscoelastic medium there is an exponential damping of the amplitude and a phase shift.

In solving the Cauchy problem we consider arbitrary initial conditions, wherein we assume that the boundary functions admit Fourier and Hankel transforms with respect to z and r, which are expandable in a Fourier series with respect to φ:

$$W_i(r, \varphi, z, t)\big|_{t=0} = f_i(r, \varphi, z) = \sum_{k=-\infty}^{\infty} e^{ik\varphi} \int_0^{\infty} J_\nu(\alpha r)\, d\alpha \int_0^{\infty} e^{i\beta z}\, \bar{f}_i^{(k)}(\alpha, \beta)\, d\beta \tag{8}$$

$$\frac{\partial}{\partial t} W_i(r, \varphi, z, t)\bigg|_{t=0} = f_{i+3}(r, \varphi, z)$$

Equating the right sides of (8) to the right sides of the solution (7) and its derivatives with respect to t at $t = 0$, we obtain the constants

$$A_i = \varphi_i^{(k)}, \quad B_i = \frac{2\lambda_i m_i \varphi_{i+3}^{(k)} + \varepsilon B_{0i} \varphi_i^{(k)}}{2\lambda_i^2 m_i - \varepsilon A_{0i}}, \quad i = 1, 2, 3$$

$$\varphi_j^{(k)} = \begin{cases} 1/2\,(\bar{f}_j^{(k)} + \bar{f}_{j+1}^{(k)}), & j = 1,4 \\ (2m_3)^{-1}\,[2\bar{f}_{j+1}^{(k)} - a_3(\bar{f}_{j-1}^{(k)} - \bar{f}_j^{(k)})], & j = 2,5 \\ (2m_2)^{-1}\,[2\bar{f}_j^{(k)} - a_2(\bar{f}_{j-2}^{(k)} - \bar{f}_{j-1}^{(k)})], & j = 3,6 \end{cases}$$

If in the expansions of the boundary functions with respect to φ we limit ourselves to a finite number of terms, the solution of the problem is then also represented in a form involving a finite number of terms and the formulas so obtained can be used for computational purposes.

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